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METHOD OF HOMOGENEOUS SOLUTIONS IN THE MIXED PROBLEM FOR A FINITE CIRCULAR CYLINDER

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The mixed axisymmetric problem of elasticity theory on the torsion of a finite circular cylinder by a stamp is considered. The stamp is fixed rigidly to one plane face of the cylinder, the other plane face is fixed, and conditions for no displacements or stresses are given on the cylinder surface. The problem is investigated by the method of homogeneous solutions [1], which permits obtaining its approximate solution for practically any values of the parameters. Such efficiency of the method is determined by the fact that the solution of the problem reduces to investigating an infinite algebraic system of the Poincaré — Koch normal systems type. When the ratio of the cylinder height to the radius of the stamp is sufficiently large, the system coefficients, the contact stresses, and the other characteristics of the problem are evaluated to any degree of accuracy, and effective asymptotic expressions are obtained for small values of this ratio. Results of numerical computations are presented.

A solution of the problem for the case of a large value of the ratio $(R - a) / h$ and small values of the ratio $\lambda = h / a$ is obtained in [2].

1. Formulation of the problem and its reduction to an infinite system. The mixed axisymmetric problem of elasticity theory on the torsion of a circular cylinder $r \leqslant R$, $0 \leqslant z \leqslant h$ (r, φ, z are cylindrical coordinates) by a stamp of radius a is considered. The stamp is rigidly adherent to the plane cylinder boundary $z = h$, while the cylinder boundary $z = 0$ and its side surface are immobile. In this case, the problem is equivalent to the following boundary value problem in the displacement function $v(r, z)$ along the φ axis:

$$\Delta v - \frac{v}{r^2} = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{1.1}$$

$$v = \delta r \quad (0 \leqslant r \leqslant a, \ z = h), \quad v = 0 \quad (r = R, \ 0 \leqslant z \leqslant h) \tag{1.2}$$

$$v = 0 \quad (z = 0, r \leqslant R), \quad \tau_{z\varphi} = G\frac{\partial v}{\partial z} = 0 \quad (a < r < R, z = h)$$

where δ is the angle of stamp rotation, and G is the shear modulus. We use the method of homogeneous solutions [1] to solve the boundary value problem (1.1) and (1.2). In conformity with the scheme of this method, we first find the solution of (1.1) when

$$\tau_{z\varphi}^{(1)} = \begin{cases} \tau(r) & (0 \leqslant r \leqslant a, \ z = h) \\ 0 & (r > a, \ z = h) \end{cases}, \quad v^{(1)} = 0 \quad (z = 0)$$

By using the Hankel integral transform we obtain ($J_1(x)$ is the Bessel function of the first kind)

$$v^{(1)}(r,z) = G^{-1}\int_0^a \tau(\rho)\rho\,d\rho \int_0^\infty \frac{\operatorname{sh} uz}{u \operatorname{ch} uh} J_1(ur) J_1(u\rho)\,u\,du \tag{1.3}$$

$$\tau^{(1)}(r,z) = \int_0^a \tau(\rho)\rho\,d\rho \int_0^\infty \frac{\operatorname{ch} uz}{\operatorname{ch} uh} J_1(ur) J_1(u\rho)\,u\,du$$

We later find the particular homogeneous solutions of (1.1) where

$$\tau_{z\varphi}^{(2)} = 0 \quad (z = h), \quad v^2 = 0 \quad (z = 0)$$

Summing the homogeneous solutions we obtain

$$v^{(2)}(r,z) = \sum_{k=1}^{\infty} B_k \operatorname{sh} u_k z J_1(u_k r) \tag{1.4}$$

$$\tau_{z\varphi}^{(2)}(r,z) = \sum_{k=1}^{\infty} B_k u_k \operatorname{ch} u_k z J_1(u_k r), \quad u_k = i\frac{\pi}{h}\left(k - \frac{1}{2}\right)$$

Using the orthogonality of the function sinh $u_k z$ in the segment $[0, h]$, we find B_k from the condition

$$v(r, z) = v^{(1)}(r, z) - v^{(2)}(r, z) = 0 \quad (r = R) \tag{1.5}$$

$$B_k = \frac{2(-1)^k K_1(R\gamma_k)}{GhI_1(R\gamma_k)}\int_0^a \tau(\rho) I_1(\rho\gamma_k)\rho\,d\rho, \quad \gamma_k = -iu_k \tag{1.6}$$

It is now easy to see that the function $v(r, z) = v^{(1)}(r, z) - v^{(2)}(r, z)$ satisfies (1.1) and all the boundary conditions (1.2) except the first. By satisfying this condition too, we obtain an integral equation in the contact stress distribution function

$$K\tau = G\delta r - G\sum_{k=1}^{\infty} B_k(-1)^{k+1} I_1(\gamma_k r) \quad (r \leqslant a) \tag{1.7}$$

$$K\tau = \int_0^a \tau(\rho)\rho\,d\rho \int_0^\infty \operatorname{th} uh J_1(ur) J_1(u\rho)\,du$$

Let us represent the solution of this equation in the form

$$\tau(r) = G\tau_0(r) - G\sum_{k=1}^{\infty} B_k(-1)^{k+1}\tau_k(r) \tag{1.8}$$

$$K\tau_k = \begin{cases} \delta r & (k = 0) \\ I_1(\gamma_k r) & (k \geqslant 1) \end{cases} \tag{1.9}$$

Then substituting (1.8) into (1.6), we obtain an infinite system of linear algebraic equations to determine the constants $B_k = x_k[hI_1(R\gamma_k)]^{-1}$:

$$x_k = g_k - \sum_{n=1} a_{kn} x_n \quad (k = 1, 2, \ldots) \tag{1.10}$$

$$g_k = 2(-1)^k K_1(R\gamma_k) T_{0,k}, \quad a_{kn} = \frac{2(-1)^{k+n+1} K_1(R\gamma_k)}{hI_1(R\gamma_n)} T_{n,k}$$

$$T_{n,k} = \int_0^a \tau_n(\rho) I_1(\rho\gamma_k) \rho\, d\rho \qquad (1.11)$$

Let us investigate the infinite system (1.10). It can be shown that

$$T_{n,k} \leqslant (2\pi a)^{-1} I_1(a\gamma_k) M_n, \quad M_n = 2\pi \int_0^a \tau_n(r) r^2\, dr$$

Let us estimate M_n $(n \geqslant 1)$. To do this we multiply both sides of (1.9) by $r\tau_0(r)$ for $k \geqslant 1$ and integrate between 0 and a. By changing the order of the integration we obtain

$$M_n = 2\pi T_{0,n} \leqslant a^{-1} I_1(a\gamma_k) M_0$$

The estimates

$$|g_k| \leqslant (\pi a)^{-1} I_1(a\gamma_k) K_1(R\gamma_k) M_0$$
$$|a_{kn}| \leqslant (\pi a^2 h)^{-1} K_1(R\gamma_k) I_1(a\gamma_k) I_1(a\gamma_n) I_1^{-1}(R\gamma_n) M_0$$

hence follow.

Taking into account the asymptotic behavior of the modified Bessel functions $K_1(x)$ and $I_1(x)$ for large values of the argument, we obtain

$$|g_k| \sim \frac{1}{2k-1} \exp\left[-\frac{\pi(R-a)}{h}\left(k-\frac{1}{2}\right)\right] \quad (k \to \infty)$$
$$|a_{kn}| \sim \frac{1}{2k-1} \exp\left[-\frac{\pi(R-a)}{h}(k+n-1)\right] \quad (k, n \to \infty)$$

We see that the free members of the system (1.10) and the coefficients of its matrix decrease exponentially as the numbers grow for $R > a$. Therefore, the system (1.10) is of the Poincaré – Koch normal system type.

Such systems also occur in investigations of certain kinds of mixed problems by the method of piecewise-homogeneous solutions [3].

To evaluate the elements of the system (1.10), the solution of the integral equations (1.9) which correspond to the fully studied contact problem on the torsion of an elastic layer by a stamp, must be known, and consequently, effective asymptotic method [4] can successfully be used to solve them.

2. Solution of the integral equation (1.9) by the method of large λ. Equation (1.9) is equivalent to the dual integral equation

$$\int_0^\infty \Phi_k(\tau) \operatorname{th} \tau\lambda J_1(\tau x)\, d\tau = f_k(x) \quad (0 \leqslant x \leqslant 1) \qquad (2.1)$$

$$\int_0^\infty \Phi_k(\tau) J_1(\tau x) \tau\, d\tau = 0 \quad (x > 1)$$

$$\Phi_k(u) = \int_0^1 \eta_k(y) J_1(uy) y\, dy, \quad \eta_k(y) = \tau_k(ya) \quad (k = 1, 2, \ldots)$$

$$f_k(x) = (\delta x, \quad \text{if} \quad k = 0; \; a^{-1} I_1(a\gamma_k x), \quad \text{if} \quad k \geqslant 1), \quad \lambda = \frac{h}{a}$$

The dual integral equation (2.1) is, in turn, equivalent to a Fredholm integral equation of the second kind [5]

$$\varphi^k(t) = \frac{1}{\pi\lambda} \int_{-1}^{1} \varphi^k(\tau) M\left(\frac{t-\tau}{\lambda}\right) d\tau = d_k(t) \quad (|t| \leqslant 1) \tag{2.2}$$

$$M(y) = \int_0^\infty [1 - \operatorname{th} u] \cos uy\, du = \sum_{k=0}^\infty b_k y^{2k} \tag{2.3}$$

$$b_k = \frac{1}{(2k)!} \int_0^\infty [1 - \operatorname{th} u] u^{2k} du = (-1)^k 2^{-2k} \sum_{i=1}^\infty i^{-2k} (-1)^{i+1}$$

$$d_k(t) = (2\delta t, \quad \text{if} \quad k = 0; \; a^{-1} \operatorname{sh}(a\gamma_k t), \quad \text{if} \quad k \geqslant 1)$$

Here

$$\eta_k(r) = \tau_k(ra) = -\frac{2}{\pi}\frac{d}{dr}\int_r^1 \frac{\varphi^k(\tau)\, d\tau}{\sqrt{\tau^2 - r^2}}, \quad \Phi_k(\tau) = \frac{2}{\pi}\int_0^1 \varphi^k(u) \sin \tau u\, du \tag{2.4}$$

In conformity with the scheme of the method of large λ (see [6], say) we seek the solution of (2.2) in the form

$$\varphi^m(t) = \sum_{n=0}^\infty \varphi_n^m(t) \lambda^{-n} \tag{2.5}$$

Substituting (2.5) and the series (2.3) into (2.2) and equating coefficients of identical powers of λ on the left and right, we obtain recurrent relationship to determine $\varphi_n{}^m(t)$

$$\varphi_0{}^m(t) = d_m(t) \tag{2.6}$$

$$\varphi_n^m(t) = \frac{1}{\pi} \sum_{k=0}^{[n-1/2]} b_k \int_{-1}^{1} \varphi_{n-2k-1}^m(\tau)(t-\tau)^{2k} d\tau \quad (n \geqslant 1)$$

(here the square brackets denote the integer part of the number). It can be seen if we follow [6], that

$$\varphi_{2s}{}^m(t) = \sum_{j=0}^{s-3} \alpha_{sj}{}^m t^{2j+1} \quad (s \geqslant 3) \tag{2.7}$$

$$\varphi_{2s+1}^m(t) = \sum_{j=0}^{s-1} \beta_{sj}{}^m t^{2j+1} \quad (s \geqslant 1), \quad \varphi_1(t) = \varphi_2(t) = \varphi_4(t) = 0$$

Now substituting (2.7) into (2.6) and equating coefficients of identical powers of t on the left and right, we obtain recurrence relationships to determine the numbers $\alpha_{sj}{}^m$ and $\beta_{sj}{}^m$:

$$\alpha_{sj}{}^{m} = -\frac{2}{\pi}\sum_{k=j+1}^{s-2} z_{kj}b_k \sum_{p=0}^{s-k-2}\frac{\beta^{m}_{s-k-1,\,p}}{2k-2j+2p+1} \quad \begin{pmatrix} s\geqslant 3 \\ 0\leqslant j\leqslant s-3\end{pmatrix} \tag{2.8}$$

$$\beta_{sj}{}^{m} = -\frac{2}{\pi} z_{sj}b_s B_{sj}{}^{m} -$$

$$\frac{2}{\pi}\sum_{k=j+1}^{s-3} z_{kj}b_k \sum_{p=0}^{s-k-3}\frac{\beta^{m}_{s-k,\,p}}{2k-2j+2p+1} \quad \begin{pmatrix} s\geqslant 1 \\ 0\leqslant j\leqslant s-1\end{pmatrix}$$

$$B_{sj}{}^{m} = \begin{cases} 2\delta\,(2s-2j+1)^{-1} & (m=0) \\ a^{-1}F^{m}_{s-j} & (m\geqslant 1)\end{cases}, \quad z_{kj} = \frac{(2k)!}{(2j+1)!\,(2k-2j-1)!}$$

$$F_n{}^{m} = \frac{(2n-1)!}{(a\gamma_m)^{2n}}\sum_{k=0}^{n-1}\left[\frac{(a\gamma_m)^{2k+1}}{(2k+1)!}\operatorname{ch} a\gamma_m - \frac{(a\gamma_m)^{2k}}{(2k)!}\operatorname{sh} a\gamma_m\right] \quad (m\geqslant 1) \tag{2.9}$$

Therefore, (2.4), (2.5), (2.7) and (2.8) permit obtaining the solution to the integral equation (1.9) to any degree of accuracy in the domain of convergence of the method of large λ proposed here, in the form of elementary and conveniently evaluated expressions

$$\varphi^{m}(t) = d_m(t) + W_m(t^{2j+1}) \tag{2.10}$$

$$W_m(t^{2j+1}) = \sum_{s=1}^{\infty}\lambda^{-2s-1}\sum_{j=0}^{s-1}\beta_{sj}{}^{m}t^{2j+1} + \sum_{s=3}^{\infty}\lambda^{-2s}\sum_{j=0}^{s-3}\alpha_{sj}{}^{m}t^{2j+1} \tag{2.11}$$

By using (1.11) and (2.4), we find

$$T_{n,k} = \frac{2a^2}{\pi}\int_0^1 \varphi^{n}(\tau)\operatorname{sh} a\gamma_k\tau\, d\tau$$

Substituting $\varphi^{n}(\tau)$ here in the form of (2.10), (2.11), we obtain

$$T_{0,k} = \frac{2a^2}{\pi}\left[2\delta F_1{}^{k} + W_0(F^{k}_{j+1})\right] \tag{2.12}$$

$$T_{n,k} = \frac{2a^2}{\pi}\left[\frac{1}{2a^2}\left(\frac{\operatorname{sh} a(\gamma_k+\gamma_n)}{\gamma_k+\gamma_n} - \frac{\operatorname{sh} a(\gamma_k-\gamma_n)}{\gamma_k-\gamma_n}\right) + W_n(F^{k}_{j+1})\right]$$

$$(n, k\geqslant 1)$$

($W_n(F_{j+1}{}^{k})$ is described by analogy with (2.11)). Now, the elements of the infinite system (1.10) are evaluated by using (2.12), and the approximate solution of this system is found by the method of reduction.

The contact stresses under the stamp $\tau_{z\varphi}(r, h) = \tau(r)$ are calculated by means of (1.8) in which if we use (2.4), (2.10), (2.11)

$$\tau_0(ra) = \frac{2}{\pi}\left[\frac{2\delta r}{\sqrt{1-r^2}} + W_0(S_j(r))\right] \tag{2.13}$$

$$\tau_k(ra) = \frac{2}{\pi}\left[-\frac{1}{a}\frac{d}{dr}\int_r^1 \frac{\operatorname{sh}(a\gamma_m\tau)\,d\tau}{\sqrt{\tau^2-r^2}} + W_k(S_j(r))\right]$$

$$S_j(r) = \frac{r}{\sqrt{1-r^2}}\, j! \sum_{k=0}^{j} \frac{r^{2(j-k-1)}}{k!\,(j-k)!} (1-r^2)^k \frac{[(2j+1)\,r^2 - 2\,(j-k)]}{2k+1}$$

($W_k\,(S_j\,(r))$ is described by analogy with (2.11)).

We find the relationship between the moment M applied to the stamp and the angle δ of stamp rotation from the condition

$$M = 2\pi \int_0^a \tau\,(r)\, r^2\, dr \tag{2.14}$$

Using (1.8), (2.4), (2.10), (2.11), we obtain

$$M = G\left[M_0 + \frac{1}{h}\sum_{k=1}^{\infty} x_k \frac{(-1)^k M_k}{I_1(R\gamma_k)}\right] \tag{2.15}$$

$$M_k = 8a^3 \int_0^1 \tau\varphi^k(\tau)\, d\tau, \quad M_k = 8a^3\left[a^{-1}F_1^{\,k} + W_k\left(\frac{1}{2j+3}\right)\right] \quad (k \geqslant 1)$$

$$M_0 = 8a^3\left[\frac{2\delta}{3} + W_0\left(\frac{1}{2j+3}\right)\right]$$

($W_k\,(1/(2j+3))$ is described by analogy with (2.11), see (2.9) for $F_1^{\,k}$).

Therefore, by using the method of large λ to solve (1.9), we have successfully obtained the solution of the problem for large values of the parameter λ in terms of the solution of the infinite system (1.10) in elementary expressions to any degree of accuracy, where the singularity in the formulas for the contact stresses is explicitly extracted. The coefficients of the infinite system are also obtained in elementary expressions.

The domain of applicability of such an approach to the solution of the problem will be examined in Sect. 4.

3. **Solution of the integral equation (1.9) by the method of small λ.** Let us examine the dual equation (2.1), which is equivalent to (1.9), and let us obtain its solution by the method of reduction to an infinite system of linear algebraic equations with a singular matrix [7]. We take the principal term in its asymptotic for small λ as the solution [8].

According to [2]

$$\tau_0(r) = \frac{\delta r}{h} + \delta \sum_{n=1}^{\infty} y_n^{\circ} I_1(\delta_n r)\, I_1^{-1}(\delta_n a), \quad \delta_n = \frac{\pi n}{h} \tag{3.1}$$

$$\tau_k(r) = \sum_{n=1}^{\infty} y_n^{\,k} I_1(\delta_n r)\, I_1^{-1}(\delta_n a) \quad (k \geqslant 1,\ r \leqslant a)$$

$Y^k = \{y_n^{\,k}\}$ $(k \geqslant 0)$ is a vector which is the solution of an infinite system of linear algebraic equations

$$BY^k = D^k, \quad B = \left\{\frac{\gamma_m K_0(\gamma_m a)\, I_1(\delta_n a) + \delta_n I_0(\delta_n a)\, K_1(\gamma_m a)}{(\delta_n^{\,2} - \gamma_m^{\,2})\, K_1(\gamma_m a)\, I_1(\delta_n a)}\right\} \tag{3.2}$$

$$D^\circ = \left\{ \frac{aK_2(\gamma_m a)}{h\gamma_m K_1(\gamma_m a)} \right\}, \quad D^k = \frac{h}{2aK_1(\gamma_k a)} \{\delta_m^k\} \quad (k \geqslant 1)$$

Here B, D^k are matrix, and vector symbols, respectively, δ_m^k is the Kronecker delta. For small λ the principal term of the asymptotic of the solution for **infinite** system is the solution of the degenerate infinite system [8]

$$AY^k = D_1^k, \; A = \{(\delta_n - \gamma_m)^{-1}\} \tag{3.3}$$

$$D_1^\circ = \left\{ \frac{a}{h\gamma_m} \right\}, \quad D_1^k = he^{\gamma_k a} \sqrt{\frac{\gamma_k}{2\pi a}} \{\delta_m^k\} \quad (k \geqslant 1)$$

The system (3.3) is obtained formally from the system (3.2) by passing to the limit as $\lambda \to 0$ $(\delta_n, \gamma_m \to \infty$ $(n, m \geqslant 1))$. The inverse A^{-1} is known for the matrix A [9]

$$A^{-1} = \{\tau_{nm}\}, \quad \tau_{nm} = \frac{(2n-1)!!\,(2m-3)!!}{h\,(2n-2)!!\,(2m-2)!!\,(2n-2m+1)}$$

and, therefore, the solution of the system (3.3), and correspondingly, the principal term of the asymptotic of the solution for the system (3.2) also become

$$y_n^\circ = \frac{a\,(2n-1)!!}{(2n)!!} \quad (n = 1, 2, \ldots) \tag{3.4}$$

$$y_n^k = e^{\gamma_k a} \sqrt{\frac{\gamma_k}{2\pi a}} \frac{(2n-1)!!\,(2k-3)!!}{(2n-2)!!\,(2k-2)!!\,(2n-2k+1)} \quad (n, k = 1, 2, \ldots)$$

The derivation of the second formula in (3.4) is evident, but the first formula in (3.4) is obtained in such a form in [2] for instance.

The magnitude of the contact stresses under the stamp and the magnitude of the moment applied to the stamp become by using (1.8), (3.1), (2.14), (2.15) and (3.4) for small λ

$$\tau(r) = \frac{G\delta}{h} \left[r + a \sum_{n=1}^{\infty} z_n \frac{I_1(\delta_n r)\,(2n-1)!!}{I_1(\delta_n a)\,(2n)!!} \right] \tag{3.5}$$

$$M = \frac{2\pi a^3 G\delta}{\lambda} \left[\frac{1}{4} + \frac{\lambda}{\pi} \sum_{n=1}^{\infty} z_n \frac{(2n-1)!!\, I_2(\delta_n a)}{n\,(2n)!!\, I_1(\delta_n a)} \right]$$

$$z_n = 1 + \frac{\pi n}{a\delta h} \sqrt{\frac{R}{a}} \sum_{k=1}^{\infty} x_k \frac{(2k-1)!!}{(2k-2)!!\,(2n-2k+1)} e^{-\gamma_k (R-a)}$$

where x_k $(k = 1, 2, \ldots)$ is the solution of the infinite system (1.10) in which by taking account of the smallness of the parameter λ, (3.1) and (3.4)

$$g_k = \delta \frac{(-1)^k 4a^2 h e^{-\gamma_k (R-a)}}{\pi^2 \sqrt{Ra}\,(2k-1)} \left[\frac{1}{2k-1} + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!\,(2m+2k-1)} \right] \tag{3.6}$$

$$a_{kn} = \frac{2\,(-1)^{k+n+1} e^{-(\gamma_k+\gamma_n)(R-a)}\,(2n-1)!!}{\pi\,(2k-1)\,(2n-2)!!} \times$$

$$\sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m-2)!!\,(2m-2n+1)\,(2m+2k-1)}$$

The smallness of the parameter $\lambda = h/a$ was used in obtaining the formulas (3.5) and (3.6), moreover, if smallness of the parameter $\varkappa = h/(R-a)$ is required then (3.5) and (3.6) can be simplified considerably for the values of the contact stresses and moment. Taking account of the relationship $\gamma_k = \pi(2k-1)(2h)^{-1}$, it can be shown that

$$z_n = 1 + \frac{\pi}{a\delta h}\sqrt{\frac{R}{a}}\frac{n}{(2n-1)}x_1 e^{-1/2\pi/\varkappa}(1 + O(e^{-\pi/\varkappa}))$$
$$x_1 = g_1(1 + O(e^{-\pi/\varkappa})) \quad \left(\varkappa = \frac{h}{R-a} \ll 1\right)$$

Substituting the value of g_1 from (3.6) in place of x_1 here, we obtain

$$z_n = 1 + \frac{4n}{\pi(2n-1)}e^{-\pi/\varkappa}\left(1 + \sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m)!!(2m+1)}\right)(1 + O(e^{-1/\varkappa})) \quad (3.7)$$

Formulas (3.5) and (3.7) yield a simple asymptotic solution of the problem for small values of the parameters λ and $\varkappa$.

Taking into account that $I_2(\delta_n a) I_1^{-1}(\delta_n a) \sim 1$ for small λ, and taking z_n in the form (3.7), the second formula in (3.5) can be simplified by summing the series. We obtain

$$M = \pi G\delta a^3\left[\frac{1}{2\lambda} + \frac{2\ln 4}{\pi} + \frac{4}{\pi}e^{-\pi(R-a)/h}(1 + O(\lambda e^{-1/\varkappa})) + O(\lambda)\right] \quad (3.8)$$

Analogously, the first formula in (3.5) is simplified for small λ and $\varkappa$ if $0 \leqslant r \leqslant a$

$$\tau(r) = G\delta\left\{\frac{r}{h} + \frac{1}{\lambda}\sqrt{\frac{a}{r}}[(1 - e^{-\pi(a-r)/h})^{-1/2} - 1 + \right. \quad (3.9)$$
$$\left. e^{-\pi(R-r)/h}(1 - e^{-\pi(a-r)/h})^{-1/2}(1 + O(\lambda e^{-1/\varkappa})) + O(\lambda)]\right\}$$

Here the singularity as $r \to a$ is separated out explicitly.

Therefore, for small values of the parameter λ two sets of formulas are obtained for the desired quantities in the problem: (3.5), (3.6), (1.10) and (3.8), (3.9). The first set is associated with the solution of the infinite system (1.10), (3.6), and the second set is not related to it but takes into account the smallness of the parameter $\varkappa$. The domain of applicability of these results is examined below.

4. **Numerical examples and analysis of the results.** Let us numerically investigate the dimensionless quantities

$$M^* = {}^3/_{16}(G\delta a^3)^{-1}M, \quad \tau^*(\rho) = (G\delta)^{-1}\tau(\rho a) \quad (0 \leqslant \rho \leqslant 1) \quad (4.1)$$

characterizing the relation between the moment applied to the stamp and the angle of stamp rotation, and the contact stresses under the stamp for different values of the parameters $\lambda = h/a$ and $R^* = R/a$. To evaluate the quantities M and $\tau(r)$ we hence use the algorithms and formulas obtained in Sects. 2 and 3.

The method of large λ (see Sect. 2) permit finding the quantities M^* and $\tau^*(\rho)$ for $\lambda \geqslant 1$ and any practically possible R^* with any degree of accuracy by using an electronic computer. The constraint on λ is associated with the convergence of series of the type (2.11).

We solve the infinite system (1.10) — (1.12) by the method of reduction. Let n_1 denote the number of equations of the system. We calculate the coefficients of this system and the quantity (4.1) to the accuracy of terms of the order of $O(\lambda^{-n_2})$. We hence select n_1 and n_2 so that the error in the final results will be exceed a given value. Let us note that the selection of the value of n_2 depends only on the parameter λ while the selection of the value of n_1 depends on the value of the quantity $(R-a)/h$. Hence, the smaller λ, the greater should n_2 be, and the smaller $(R-a)/h$, the greater should n_1 be.

For instance, 0.01% accuracy is achieved for $\lambda=1$ with $n_2=60$, for $\lambda=1.5$ and $n_2=10$, while a 1% accuracy is achieved for $\lambda=1$ with $n_2=20$. In order for the error not to exceed 0.1% for $(R-a)/h>0.1$ it is necessary to take $n_1 \leqslant 5$, although the error will not exceed 0.0003% for $(R-a)/h>0.2$ and $n_1=5$. A 1% accuracy is achieved for $(R-a)/h=0.0007$ if $n_1=30$, and $n_1=5$ for $(R-a)/h=0.07$.

The use of the method of small λ (see Sect. 3) permits finding the approximate value of the quantities M^* and $\tau^*(\rho)$ by means of (3.5), (3.6) and (1.10) by using an electronic computer with not more than a 3% error for $\lambda \leqslant 0.5$. If $0.5<\lambda<1$, then the solution of the system (3.2) must be found asymptotically by the method of reduction without simplifying it, as was done in [2], in order to evaluate the coefficients of the infinite system (1.10). The above is valid here relative to the selection of n_1.

Formulas (3.8) and (3.9) permit the evaluation of M^* and $\tau^*(\rho)$ with an error not exceeding 3% for $\lambda<0.5$ and $(R-a)/h>0.5$.

Values of the quantities M^* and $\tau^*(\rho)$ are presented below for different values of the parameters ρ, λ and R^*

$R^*=1.3;\ 1.5$	1.3	1.5	1.5	1.5	1.5
$\lambda=0.1$	1.5	0.3	1.0	1.0[2]	1.5
$\tau^*(0.1)=1.000$	0.1492	0.3336	0.1469	0.1457	0.1417
$\tau^*(0.3)=3.000$	0.4689	1.002	0.4572	—	0.4442
$\tau^*(0.5)=5.000$	0.8697	1.679	0.8380	0.8336	0.8188
$\tau^*(0.7)=7.000$	1.503	2.422	1.422	—	1.400
$\tau^*(0.9)=9.235$	3.266	3.853	3.007	2.983	2.983
$\tau^*(0.95)=10.77$	4.864	5.117	4.441	—	4.412
$M^*=3.448$	1.244	1.464	1.145	1.021	1.135
$R^*=1.01$	1.01	1.05	1.05	1.1	1.1
$\lambda=0.1$	1.0	0.1	1.5	0.1	1.5
$\tau^*(0.1)=1.000$	0.1782	1.000	0.1709	1.000	0.1646
$\tau^*(0.3)=3.000$	0.5689	3.000	0.5437	3.000	0.5258
$\tau^*(0.5)=5.000$	1.097	5.000	1.038	5.000	0.9913
$\tau^*(0.7)=7.001$	2.082	7.001	1.911	7.001	1.778
$\tau^*(0.9)=9.606$	6.068	9.336	4.896	9.255	4.220
$\tau^*(0.95)=12.79$	11.09	11.29	7.966	10.87	6.524
$M^*=4.131$	2.940	3.606	1.973	3.478	1.627

For $\lambda \geqslant 1$, the results of Sect. 2 are used when the solution of the integral equation (1.9) was found by the method of large λ, in this case the values of M^* and $\tau^*(\rho)$ are presented to four-figure accuracy. For $\lambda < 1$ the results of Sect. 3 are used. In this case the smaller λ, the more accurate the result.

The numerical results are in good agreement with the results in [2].

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